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Chaotic Root-finding for a Small Class of Polynomials

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In this paper we present a new closed-form solution to a chaotic difference equation, $y_{n+1} = a_2 y_n^2 + a_1 y_n + a_0$ with coefficient $a_0 = (a_1 - 4)(a_1 + 2)/(4a_2)$, and using this solution, show how corresponding exact roots to a special set of related polynomials of order 2^p, $p \in \mathbb{N}$ with two independent parameters can be generated, for any p.

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INTRODUCTION

The difference equation $y_{n+1} = f(y_n)$, $n = 0, 1, 2\ldots$ where

$$f(y) = a_2 y^2 + a_1 y + \frac{(a_1 - 4)(a_1 + 2)}{4a_2} \tag{1}$$

and $a_1, a_2 \in \mathbb{C}$, $a_2 \neq 0$ has the exact, general solution

$$y_n(\omega) = \frac{1}{a_2}\left(2\cos(\omega 2^n) - \frac{a_1}{2}\right). \tag{2}$$

We discovered this solution from the solution presented in Ref. [5] by reparametrising with two new variables as, for example, $y_n = c\cos(\omega 2^n) + d$. We then inserted this into the difference equation and equated powers of $\cos(\omega 2^n)$, obtaining that $c = 2/a_2$ and $d = -a_1/2a_2$. We were also interested to notice that the fixed-point problem for Eq. (1) is also a root-finding problem for the polynomial

$$f^p(y) - y = 0 \tag{3}$$

The result known as *Abel's impossibility theorem* states that there are no exact expressions for finding the roots of general polynomials of order greater than four in terms of a finite number of elementary operations. Therefore, in general, we would be forced to use an iterative root-finding method for this polynomial (see, for example, Refs. [1,2]). These methods are typically difference equations, for example, the method of Newton–Raphson

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iteration for solving the equation $F(y) = 0$ is, given a close guess y_0:

$$y_{n+1} = y_n - \frac{F(y_n)}{F'(y_n)} \tag{4}$$

The main requirement of this method is convergence to some fixed point. However, a problem sometimes arises in that the system (4) may well oscillate, or in some cases, behave chaotically. Much progress has been made by numerical analysts in proving convergence and inventing better methods that are stable even for bad initial guess values, see for example, Ref. [3]. Some beautiful methods have been devised that combine numerics with topology. They make it possible to find approximate solutions of a given system by a continuous deformation of the solutions of a related one that is exactly solvable, guaranteeing global convergence [4]. However, in this case, as in other special cases, we can find the roots of the polynomial (3) exactly, without the need for iteration. The rest of the paper shows how to do this explicitly.

PERIODIC ORBITS ARE ROOTS OF THE POLYNOMIAL

The impossibility theorem is a general one: there are rare and special cases where it does not hold, and this paper presents one of these special cases, using a result from chaotic dynamics. The difference system (1) has two fixed points. It also has a countable infinity of periodic orbits of all cycle lengths. Rearranging the periodic orbit equation gives Eq. (3) which is a polynomial equation of order 2^p, the first two of which are:

$$a_2y^2 + (a_1 - 1)y + \frac{(a_1 - 4)(a_1 + 2)}{4a_2} = 0$$

and

$$a_2^3y^4 + 2a_2^2a_1y^3 + a_2\left(\frac{3}{2}a_1^2 - 4\right)y^2 + \left(\frac{1}{2}a_1^3 - 4a_1 - 1\right)y + \frac{1}{a_2}\left(\frac{1}{16}a_1^4 - a_1^2 + \frac{1}{2}a_1 + 2\right)$$
$$= 0.$$

Thus, in finding the periodic points of the equation $f^p(y) = y$, we also find exact solutions to the polynomial equation (3). We shall now show the technical details involved in finding these solutions.

Since the cosine function is bounded, the solution (2) is also bounded. In addition, as n increases, the binary expansion of the expression $\omega 2^n$ shifts successively leftward, as described in Ref. [5], and many other texts on chaotic dynamical systems (see, for example, Ref. [6] or Ref. [7]). A key result is that if the binary digit expansion of $\omega/2\pi$ is periodic, then the behaviour of the solution is also periodic. Thus, finding the $\omega/2\pi$ that have periodic binary expansions leads us to the periodic points of $f^p(y) = y$, which in turn, can be interpreted as the roots of the 2^p order polynomial.

Therefore, for the purpose of this paper, only values of

$$\phi = \omega/2\pi$$

that have periodic binary digit expansions are relevant to us. Since irrational numbers are not periodic, we must choose ϕ rational, i.e. we want $\phi = k/L$ with $k, L \in \mathbb{N}$ such that ϕ has

GDEA 41040—14/7/2004——112945

a periodic binary digit expansion, with period $p = \log_2(m)$, where m is the order of the polynomial for which we wish to find the roots.

CONSTRUCTING ϕ

Rational fractions with periodic binary digit expansions with periodicity p correspond to the following expression:

$$\phi = \sum_{m=1}^{\infty} k(2^p)^{-m} = \frac{k}{2^p - 1}.$$

As mentioned in the previous section, the periodicity of the binary expansion of ϕ implies periodicity of the solutions to Eq. (3). To see this, note that:

$$\begin{aligned} 2^p\phi = 2^p\left(\sum_{m=1}^{\infty} k(2^p)^{-m}\right) &= \sum_{m=1}^{\infty}(2^p)k(2^p)^{-m} = \sum_{m=1}^{\infty} k(2^p)^{(-m+1)} \\ = \sum_{m=0}^{\infty} k(2^p)^{-m} = k + \sum_{m=1}^{\infty} k(2^p)^{-m} &\equiv \sum_{m=1}^{\infty} k(2^p)^{-m} \equiv \phi \pmod 1) \end{aligned}$$

Therefore, $2^p\phi \equiv \phi \pmod 1$, which implies, since $\omega = 2\pi\phi$, that $2^p\omega \equiv \omega \pmod{2\pi}$, and we reach the conclusion that $\cos(2^p\omega) = \cos(\omega)$, as required.

There is, however, a minor complication to this scheme due to the symmetry of the cosine function about π in the general solution (2). This symmetry implies that values of ω that lie equidistant from π produce the same solution. Then

$$|\omega - \pi| = |\phi 2\pi - \pi| = \pi\left|\frac{2k}{2^p - 1} - 1\right| = \frac{\pi}{2^p - 1}|(2k+1) - 2^p|$$

must be unique for all choices of k. However,

$$|(2n+1) - 2^p| = |2^p - (2n+1)|, \tag{5}$$

so that $k = m$ and $k = 2^p - (m+1)$ for $m = 0, 1 \ldots 2^p - 1$ lead to symmetrically identical solutions to the polynomial. Therefore, only $k = 0, 1 \ldots 2^{p-1} - 1$ give the unique required solutions.

The consequence of this is that in order to find all solutions of the 2^p order polynomial, we must seek solutions to the polynomial of order 2^{2p} instead, since all periodic points of $f^p(y) = y$ also satisfy $f^{2p}(y) = y$.

We therefore set $L = 2^{2p} - 1$. However, the converse is not true: not all the periodic points of $f^{2p}(y) = y$ are periodic points of $f^p(y) = y$. Therefore, in enumerating the solutions of the polynomial, ϕ must satisfy two criteria:

(i) As mentioned above, the symmetry of the cos function implies that ϕ must be less than (1/2), which in turn implies that $k < 2^{2p-1}$

(ii) ϕ must either have a periodic binary digit expansion with period p, or period $2p$ but have ω symmetric about π under the iterative shift of p digits, i.e. ϕ and $\phi 2^p$ lie equidistant from π.

Therefore, to construct and enumerate all ϕ with binary digit expansions of period p when embedded in a sequence of $2p$, we set:

$$k = m + m2^p = m(2^p + 1)$$

for $m = 0, 1 \ldots 2^{p-1} - 1$. We thus enumerate half of the required solutions. Secondly, from Eq. (5), to find all the symmetric ϕ with binary digit expansions of period 2_p, we then choose:

$$k = m2^p - m = m(2^p - 1)$$

for $m = 1, 2 \ldots 2^{p-1}$, and we have thereby enumerated the remaining solutions.

A HIGH-ORDER EXAMPLE

Here we demonstrate an application of the method to the order 8 polynomial $f^3(y) - y = 0$. The following expressions for the coefficients in ascending order of y are:

$$y^0 : \frac{1}{a_2}\left(2 - \frac{1}{2}a_1 - 4a_1^2 + \frac{5}{4}a_1^4 - \frac{1}{8}a_1^6 + \frac{1}{256}a_1^8\right)$$

$$y^1 : -1 - 16a_1 + 10a_1^3 - \frac{3}{2}a_1^5 + \frac{1}{16}a_1^7$$

$$y^2 : a_2\left(-16 + 30a_1^2 - \frac{15}{2}a_1^4 + \frac{7}{16}a_1^6\right)$$

$$y^3 : a_2^2\left(40a_1 - 20a_1^3 + \frac{7}{4}a_1^5\right)$$

$$y^4 : a_2^3\left(20 - 30a_1^2 + \frac{35}{8}a_1^4\right)$$

$$y^5 : a_2^4(-24a_1 + 7a_1^3)$$

$$y^6 : a_2^5(-8 + 7a_1^2)$$

$$y^7 : 4a_1a_2^6$$

$$y^8 : a_2^7$$

We use the LaGuerre root-finding method [2] with $a_2 = -1 + i, a_1 = 2 - i$, and we find that numerical solutions to the polynomial accurate to four decimal places are:

$$y = \{-0.25 - 0.75i,\ -0.016 - 0.516i,\ 0.1265 - 0.3735i,\ 0.5764 + 0.0764i,\ 0.97252 + 0.4725i,\ 1.25 + 0.75i,\ 1.651 + 1.151i,\ 1.6897 + 1.1897i\}$$

For this polynomial, $p = 3$ and so $L = 2^{2p} - 1 = 63$. Next, we enumerate the solutions, firstly for $m = 0, 1 \ldots 2^{p-1} - 1 = 3$. Therefore,

$$k = m(2^p + 1) = 0, 9, 18, 27$$

$$\phi_m = \left\{\frac{0}{63}, \frac{9}{63}, \frac{18}{63}, \frac{27}{63}\right\} = \left\{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}\right\}$$

or in binary notation

$$\phi_m = \{0.\overline{000000}, 0.\overline{001001}, 0.\overline{010010}, 0.\overline{011011}\}$$

where for example $0.\overline{010010}$ indicates the infinite binary digit repetition of the sequence 010010.

From these values of ϕ_m we then calculate the first half set of solutions, here given to eight decimal places:

$$y_m = \frac{1}{a_2}\left(2\cos(\phi_m 2\pi) - \frac{a_1}{2}\right)$$

$$= \{-0.25 - 0.75i,\ 0.1265102 - 0.3734898i,\ 0.97252093 + 0.47252093i,\ 1.65096887 + 1.15096887i\}$$

Secondly, we enumerate the symmetric set for $m = 1, 2 \ldots 2^{p-1}$:

$$k = m(2^p - 1) = \{7, 14, 21, 28\}$$

$$\phi_m = \left\{\frac{7}{63}, \frac{14}{63}, \frac{21}{63}, \frac{28}{63}\right\} = \left\{\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}\right\}$$

$$= \{0.\overline{000111},\ 0.\overline{001110},\ 0.\overline{010101},\ 0.\overline{011100}\}$$

from which we can calculate the second set of solutions, again to eight decimal places:

$$y_m = \frac{1}{a^2}\left(2\cos(\phi_m 2\pi) - \frac{a_1}{2}\right)$$

$$= \{-0.01604444 - 0.51604444i,\ 0.57635182 + 0.07635182i,\ 1.25 + 0.75i,\ 1.68969262 + 1.18969262i\}$$

CONCLUSIONS

By finding a general solution to a discrete, chaotic system and interpreting a problem of finding roots of a high-order polynomial as the fixed point equation for that chaotic system, we have shown how to obtain exact roots to the polynomial. We then compared this with the results of a numerical root-finding method. Of course, we have not presented a general root-finding method, but we find it intriguing to notice that iterative root-finding methods are often difference equations in their own right, for which general solutions of particular cases may well be known.

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